Mass distribution of highly flattened galaxies and modified Newtonian dynamics

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Dynamics of spiral galaxies derived from a given surface mass density has been derived earlier in a classic paper. We try to transform the singular elliptic function in the integral into a compact integral with regular elliptic function. Solvable models are also considered as expansion basis for RC data. The result makes corresponding numerical evaluations easier and analytic analysis possible. It is applied to the study of the dynamics of Newtonian system and MOND as well. Careful treatment is shown to be important in dealing with the cut-off of the input data.

Keywords: dark matter; galaxies: kinematics and dynamics

I. INTRODUCTION

The rotation curve (RC) observations indicates that less than 10 % of the gravitational mass can be measured from the luminous part of spiral galaxies. This is the first evidence calling for the existence of un-known dark matter and dark energy. In the meantime, an alternative approach, Modified Newtonian dynamics (MOND) proposed by Milgrom [14], has been shown to agree with many rotation curve observations [14, 15, 17, 19, 20, 21].

Milgrom argues that dark matter is redundant in the approach of MOND. The missing part was, instead, proposed to be derived from the conjecture that gravitational field deviates from the Newtonian $1/r^2$ form when the field strength g is weaker than a critical value $g_0 \sim 0.9 \times 10^{-8}$ cm s^{-2} [21].

In MOND the gravitational field is related to the Newtonian gravitational field g_N by the following relation:

$$g \cdot \mu_0(\frac{g}{g_0}) = g_N \tag{1}$$

with a function μ_0 considered as a modified inertial. Milgrom shows that the model with

$$\mu_0(x) = \frac{x}{\sqrt{1+x^2}} \tag{2}$$

agrees with RC data of many spiral galaxies [9, 14, 15, 17, 19, 20, 21]. The alternative theory could be compatible with the spatial inhomogeneity of general relativity theory. Various approaches to derive the collective effect of MOND has been an active research interest recently. [10, 22]

Recently, it was shown that a simpler inertial function of the following form [6, 26]

$$\mu_0(x) = \frac{x}{1+x} \tag{3}$$

fits better with RC data of Milky Way and NGC3198. Indeed, one can show that the MOND field strength g is, from Eq. (1),

$$g = \frac{\sqrt{{g_N}^2 + 4g_0 g_N} + g_N}{2}. \tag{4}$$

The model shown by Eq. (2) will be denoted as Milgrom model, while the model (3) will be denoted as FB model. We will study and compare the results of these two models in the paper.

Accumulated evidences show that the theory of MOND is telling us a very important message. Either the Newtonian force laws do require modification in the weak field limit or the theory of MOND may just represent some collective effect of the cosmic dark matter awaiting for discovery. In both cases, the theory of MOND deserves more attention in order to reveal the complete underlying physics hinted by these successful fitting results. Known problem with momentum conservation has been resolved with alternative covariant theories[2, 7, 10, 11, 13, 18, 22].

It was also pointed out that the proposed inertial function μ_0 could be functional of the whole N field and could be more complicate than the ones shown earlier [4, 5, 16]. Successful fitting with the rotation curves of many existing spiral galaxies indicates, however, that the most important physics probably has been revealed by these simple inertial functions shown in this paper. One probably should take these models more seriously in order to generate more clues to the final theory.

If the theory of the MOND is the final theory of gravitation without any dark matter in a large scale system with some inertial function $\mu_0(x)$, one should be able to derive the precise mass distribution form the measured RC

provided that the distance D is known. Earlier on, the mass distribution derived from the rotation curve measurement can only be used to predict how much dark matter is required in order to secure the Newtonian force law. In the case of MOND, one should be able to plot a dynamical profile of the Γ function $\Gamma(r) \equiv M(r)/L(r)$ with the measured luminosity function $L(r)$ following the theory of MOND. The Γ function should then provide us useful information about the detailed distribution of stars with different luminous spectrum obeying the well-known $L \sim M^\alpha$ relation.

The dynamical profile $\Gamma(r)$ can be treated as important information regarding the detailed distribution of stars with different mass-luminosity relations within each spiral galaxy[8]. Even one still does not know a reliable way to derive the inertial function $\mu_0(x)$, there are successful models shown earlier as useful candidates for the theory of MOND. It would be interesting to investigate the dependence of these models with the M/L profiles. Hopefully, information from these comparisons will provide us clues to the final theory.

Therefore, we will try first to derive the mass density for Milgrom and Famaey&Binney (FB) models in details. There are certain boundary constraints needed to be relaxed for the asymptotic flat rotation curve boundary condition which is treated differently in the original derivation of these formulae [23]. It is also important to compare the effects of exterior contribution between the Milgrom and FB models for a more precise test of the fitting application.

We will first review briefly how to obtain the surface mass density $\mu(r)$ from a given Newtonian gravitational field g_N with the help of the elliptic function $K(r)$ in section II. The integral involving the Bessel functions is derived in detailed for heuristical reasons in this section too.

In addition, a series of integrable model in the case of Newtonian model, Milgrom model as well as the FB model for MOND will be presented in section III. These solvable models will be shown to be good expansion basis for the RC curve data for spiral galaxies. In practice, this expansion method will help us better understand the analytic properties of the spiral galaxies.

We will also try to convert the formula shown in section II into a simpler form making numerical integration more accessible in section IV. The apparently singular elliptic function $K(r)$ is also converted to combinations of regular elliptic function $E(r)$ by properly managed integration-by-part.

In section V, one derives the interior mass contribution $\mu(r < R)$ from the possibly unreliable data $v(r > R)$ both in the cases of MOND and in the Newtonian dynamics. The singularity embedded in the useful formula is taken care of with great caution. Similarly, one tries to derive the formulae related g_N from a given $\mu(r)$ in section VI. One also presents a simple model of exterior mass density $\mu(r > R)$ in this section. The corresponding result in the theory of MOND is also presented in this paper. Finally, we draw some concluding remarks in section VII.

II. NEWTONIAN DYNAMICS OF A HIGHLY FLATTENED GALAXY

Given the surface mass density $\mu(r)$ of a flattened spiral galaxy, one can perform the Fourier-Bessel transform to convert $\mu(r)$ to $\mu(k)$ in k-space via the following equations [1]

$$\mu(r) = \int_0^\infty kdk\; \mu(k) J_0(kr), \tag{5}$$

$$\mu(k) = \int_0^\infty rdr\; \mu(r) J_0(kr) \tag{6}$$

with $J_m(x)$ the Bessel functions. Note that the closure relation

$$\int_0^\infty kdk J_m(kx) J_m(kx') = \frac{1}{x}\delta(x - x') \tag{7}$$

can be used to convert Eq. (6) to Eq. (5) and vice versa with a similar formula integrating over dr. The Green function of the equation

$$\nabla^2 G(x) = -4\pi G \delta(r)\delta(z) \tag{8}$$

can be read off from the identity

$$\frac{1}{\sqrt{r^2 + z^2}} = \int_0^\infty dk\; \exp[-k|z|] J_0(kr). \tag{9}$$

Therefore, the Newtonian potential ϕ_N can be shown to be [12, 23]

$$\phi_N(r, z) = 2\pi G \int_0^\infty dk\; \mu(k) J_0(kr) \exp[-k|z|] \tag{10}$$

with a given surface mass density $\mu(r)$. It follows that

$$g_N(r) = -\partial_r \phi_N(r, z=0) = 2\pi G \int_0^\infty kdk \; \mu(k) J_1(kr). \tag{11}$$

The Newtonian gravitational field $g_N(r) = v_N^2(r)/r$ can be readily derived with a given function of rotation velocity $v_N(r)$. Here one has used the recurrence relation $J_1(x) = -J_0'(x)$ in deriving Eq. (11). Furthermore, from the closure relation (7) of Bessel function, one can show that the function $g_N(r)$ satisfies the following conversion equation

$$g_N(r) = \int_0^\infty kdk \int_0^\infty r'dr' \; g_N(r') J_1(kr) J_1(kr'). \tag{12}$$

Therefore, one has

$$\mu(k) = \frac{1}{2\pi G} \int_0^\infty rdr \; g_N(r) J_1(kr). \tag{13}$$

Hence, with a given surface mass density $\mu(r)$ for a flattened spiral galaxy, one can show that

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty kdk \int_0^\infty dr' \; r' g_N(r') J_0(kr) J_1(kr'). \tag{14}$$

Assuming that $\lim_{r\to 0} r g_N(r) \to 0$ and $\lim_{r\to\infty} r g_N(r) < \infty$ hold as the boundary conditions, one can perform an integration-by-part and show that

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty dk \int_0^\infty dr' \; \partial_{r'}[r' g_N(r')] J_0(kr) J_0(kr') \tag{15}$$

with the help of the asymptotic property of Bessel function: $J_m(r \to \infty) \to 0$. Here we have used the recurrence relation $J_1(x) = -J_0'(x)$ in deriving above equation.

Note that the boundary terms can be eliminated under the prescribed boundary conditions. In fact, the limit $\lim_{r\to\infty} r g_N(r) < \infty$ is slightly different from the original boundary conditions given in [23]. The difference is aimed to make the system consistent with the flatten RC measurement which implies that $\lim_{r\to\infty} r g_N(r) = v_N^2(r \to \infty) \to$ constant $< \infty$. In summary, one only needs to modify the asymptotic boundary condition in order to eliminate the surface term during the process of integration-by-part. We have relaxed this boundary condition to accommodate any system with a flat rotation curve.

One can further define

$$H(r, r') = \int_0^\infty dk J_0(kr) J_0(kr') \tag{16}$$

and write the function $\mu(r)$ as

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty dr' \; \partial_{r'}[r' g_N(r')] H(r, r'). \tag{17}$$

Note that the function $H(r, r')$ can be shown to be proportional to the elliptic function $K(x)$:

$$H(r, r') = \frac{2}{\pi r_>} K(\frac{r_<}{r_>}) \tag{18}$$

with $r_>$ ($r_<$) the larger (smaller) of r and r'.

The proof is quite straightforward. Since some properties of the Bessel functions are very important in the dynamics of the spiral disk, as well as many disk-like system, we will show briefly the proof for heuristical reason. One of the purpose of this derivation is to clarify that there are different definitions for the elliptic functions written in different textbooks. Confusion may arise applying the formulae in a wrong way.

Note first that the Bessel function $J_0(x)$ has an integral representation [1, 25]

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos[x \sin\theta] d\theta. \tag{19}$$

Given the delta function represented by the plane wave expansion:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp[ikx], \tag{20}$$

One can show that, with the integral representation of J_0,

$$\int_0^\infty dx \cos[kx]\ J_0(x) = \frac{1}{\sqrt{1-k^2}} \tag{21}$$

for $k < 1$. On the contrary, above integral vanishes for $k > 1$. Therefore, one can apply above equation to show that

$$\int_0^\infty dk J_0(kx) J_0(k) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \frac{2}{\pi}K(x) \tag{22}$$

with the help of the Eq. (19) again. The last equality in above equation follows exactly from the definition of the elliptic function K.

There is an important remark here. Note that the elliptic functions $E(k)$ and $K(k)$ used in this paper, and Ref. [1, 23] as well, are defined as

$$K(x) \equiv \int_0^{\pi/2} (1-x^2\sin^2\theta)^{-1/2} d\theta, \tag{23}$$

$$E(x) \equiv \int_0^{\pi/2} (1-x^2\sin^2\theta)^{1/2} d\theta \tag{24}$$

which is different from certain textbooks. Some textbooks, and similarly some computer programs, prefer to define above integrals as $K(x^2)$ and $E(x^2)$ instead. In fact, one can check the differential equations satisfied by $E(x)$ and $K(x)$ that will be shown explicitly shortly in next section. Before adopting the equations presented in the texts, one should check the definition of these elliptic functions carefully for consistency.

After a proper redefinition of $k \to k'r'$ and write $x = r/r' < 1$, one can readily prove that the assertion (16) is correct.

Note that the series expansion of the elliptic function $K(x)$ is

$$K(x) = \frac{\pi}{2}\sum_{n=0}^{\infty} C_n^2 x^{2n} \tag{25}$$

with $C_0 = 1$ and $C_n = (2n-1)!!/2n!!$ for all $n \geq 1$. Therefore, one can show that it agrees with the series expansion of the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}, 1; \frac{r_<^2}{r_>^2})$, up to a factor $\pi/2$. Indeed, one has:

$$F(a,b,c;x^2) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^{2n}, \tag{26}$$

with $(a)_n \equiv \Gamma(n+a)/\Gamma(a)$. It is straightforward to show that the expansion coefficients of Eq.s (25) and (26) agree term by term. Therefore, the derivation shown above agrees with the result shown in [23, 25]:

$$H(r,r') = \frac{1}{r_>}F(\frac{1}{2}, \frac{1}{2}, 1; \frac{r_<^2}{r_>^2}) = \frac{2}{\pi r_>}K(\frac{r_<}{r_>}). \tag{27}$$

with $r_>$ ($r_<$) the larger (smaller) of r and r'.

As a result, one has

$$\mu(r) = \frac{1}{2\pi G}\int_0^\infty \partial_{r'}[v_N^2(r')]\ H(r,r') dr' \tag{28}$$

given the identification $g_N = v_N^2/r$.

III. SOME SIMPLE INTEGRABLE MODELS

One can show that $\mu(r)$ is integrable with a Newtonian velocity described by

$$v_N^2(r) = \frac{C_0^2 a}{\sqrt{r^2+a^2}}, \tag{29}$$

with C_0 and a some constants of parametrization. Note that the velocity function $v_N(r)$ vanishes at spatial infinity while $v_N(0) \to C_0$ with a non-vanishing value. One notes that the observed velocity at $r=0$ is expected to be zero from symmetric considerations. We will come back to this point later for a resolution. Following Eq. (14), one can write

$$\mu(r) = \frac{1}{2\pi G}\int_0^\infty kdk\Lambda(k)J_0(kr) \tag{30}$$

with $\Lambda(k)$ defined as

$$\Lambda(k) \equiv \int_0^\infty \frac{C_0^2 a}{\sqrt{r^2+a^2}} J_1(kr)dr. \tag{31}$$

To evaluate $\Lambda(k)$, one will need the following formula:

$$\int_0^\infty dk \exp[-kx]J_0(k) = \frac{1}{\sqrt{1+x^2}} \tag{32}$$

which follows from Eq. (21) by replacing $x \to ix$ with the help of an analytic continuation. In addition, writing $\exp[-kx]\, dk$ as $-d(\exp[-kx])/x$ and performing an integration-by-part, one can derive

$$\int_0^\infty dk \exp[-kx]J_1(k) = 1 - \frac{x}{\sqrt{1+x^2}}. \tag{33}$$

Here we have used the identity $J_0' = -J_1$ and the fact that $J_0(0) = 1$. With the help of above equation, one can show that

$$\int_0^\infty dk(1-\exp[-2k])J_1(kx) = \frac{2}{x\sqrt{x^2+4}}. \tag{34}$$

Multiplying both sides of above equation with $J_1(k'x)xdx$, one can integrate above equation and obtain

$$\int_0^\infty dx J_1(kx)\frac{2}{\sqrt{x^2+4}} = \frac{1-\exp[-2k]}{k} \tag{35}$$

with the help of the closure relation (7). After a redefinition of parameters $k \to ka/2$ and $x \to 2r/a$, one can show that

$$\int_0^\infty dr J_1(kr)\frac{a}{\sqrt{r^2+a^2}} = \frac{1-\exp[-ak]}{k} \tag{36}$$

and hence

$$\Lambda(k) = \frac{C_0^2}{k}\left(1-\exp[-ak]\right). \tag{37}$$

In additon, equation (32) can also be written as

$$\int_0^\infty dk \exp[-kx]J_0(ka) = \frac{1}{\sqrt{x^2+a^2}} \tag{38}$$

after proper reparametrization. Therefore, one has

$$\begin{aligned}\mu(r) &= \frac{C_0^2}{2\pi G}\int_0^\infty dk(1-\exp[-ak])J_0(kr) \\ &= \frac{C_0^2}{2\pi G}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+a^2}}\right].\end{aligned} \tag{39}$$

Hence this model is integrable as promised. Also, as mentioned earlier in this section, this model with a small C_0 and properly adjusted a can describe the velocity profile pretty nice in the case of MOND.

One can eliminate the non-vanishing constant by two different methods. The first method is simply subtracting two integrable $v_N^2(r, C_0, a_i)$, namely, define the new Newtonian velocity as

$$v_{N0}^2(r) = C_0^2 \left(\frac{a_1}{\sqrt{r^2 + a_1^2}} - \frac{a_2}{\sqrt{r^2 + a_2^2}} \right), \tag{40}$$

with C_0 and $a_1 > a_2$ some constants of parameterizations. This new velocity function is also integrable due to the linear dependence of the function $v_N^2(r)$ in Eq. (28). In addition, it vanishes at $r = 0$ and approaches 0 at spatial infinity $r \to \infty$. Therefore, one has

$$\mu_0(r) \;=\; \frac{C_0^2}{2\pi G} \left[\frac{1}{\sqrt{r^2 + a_2^2}} - \frac{1}{\sqrt{r^2 + a_1^2}} \right] \tag{41}$$

directly from Eq. (39). In addition, one can also show that [23] higher derivative models defined by

$$v_{Nn}^2(r) = -C_n^2 (-\frac{\partial}{\partial a^2})^n \frac{a}{\sqrt{r^2 + a^2}} = \sum_{k=1}^{n} \frac{C_k^n (2k-1)!(2n-2k)!}{2^{2n-1}(k-1)!(n-k)!(2k-1)} a^{1-2k} (r^2+a^2)^{-n+k-1/2} - \frac{(2n)!}{2^{2n} n!} a (r^2+a^2)^{-n+1/2} \tag{42}$$

are also integrable and give the mass density as

$$\mu_n(r) \;=\; -\frac{C_n^2}{2\pi G} (-\frac{\partial}{\partial a^2})^n \left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + a^2}} \right] = \frac{C_n^2}{2\pi G} (-\frac{\partial}{\partial a^2})^n \left[\frac{1}{\sqrt{r^2 + a^2}} \right] = \frac{C_n^2}{2\pi G} \frac{(2n)!}{2^{2n} n!} (r^2 + a^2)^{-n-1/2}. \tag{43}$$

Mote that both v_{Nn}^2 and $\mu_n(r)$ are in fact functions of $(r^2 + a^2)^{-n-1/2}$ with appropriate combinations. For example, given the Newtonian velocity

$$v_{N1}^2(r) = \frac{C_1^2 r^2}{a(r^2 + a^2)^{3/2}} = C_1^2 \left[\frac{1}{a(r^2 + a^2)^{1/2}} - \frac{a}{(r^2 + a^2)^{3/2}} \right], \tag{44}$$

the corresponding mass density will be given by

$$\mu_1(r) = \frac{C_1^2}{2\pi G (r^2 + a^2)^{3/2}}. \tag{45}$$

For convenience, we have absorbed a common factor 1/2 into C_1^2. Note that $v_{N1}^2(r \to \infty) \to 0$ and $v_1(r = 0) = 0$ in this model.

A. Newtonian Model

Consider the case of Newtonian model that observed velocity v and Newtonian velocity v_N are identical. This model is known to require the presence of dark matters[27]. Since the velocity has to vanish at $r = 0$ and goes to a constant at spatial infinity, one will show that an additional constant term added to the v_N^2 will provide both resolution at the same time. Indeed, another way to eliminate the non-vanishing velocity at $r = 0$ is to introduce a constant velocity by noting that

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty k dk \Lambda(k) J_0(kr) = \frac{C_0^2}{2\pi G r} \tag{46}$$

with $\Lambda(k)$ defined as

$$\Lambda(k) \equiv C_0^2 \int_0^\infty J_1(kr) dr = \frac{C_0^2}{k}. \tag{47}$$

Here we have used the identity

$$\int_0^\infty dk J_n(kr) = \frac{1}{r} \tag{48}$$

which follows directly from Eq. (32-33).

Therefore, one can show that the Newtonian velocity given by the form

$$v_0^2(r) = v_{N0}^2(r) = C_0^2 \left[1 - \frac{a}{\sqrt{r^2 + a^2}} \right] \tag{49}$$

is induced by the mass density of the following form

$$\mu_0(r) \;=\; \frac{C_0^2}{2\pi G} \frac{1}{\sqrt{r^2 + a^2}}. \tag{50}$$

Here C_0 and a are some constants of parametrization. Note that the Newtonian velocity (49) vanishes at $r = 0$ and $V_N \to C_0$ at spatial infinity as promised. Therefore, this velocity profile goes along with the observed RC of the spirals. Hence this model can be used to simulate the dynamics of spiral galaxies which requires the presence of dark matters. Hence we will call this model with $v = v_N$is the Newtonian model that normally requires the existence of dark matters.

This zero mode is however the only known integrable modes for velocity profiles. Successive differentiating v_N^2 in this model simply turns off the constant term C_0^2 in Eq. (49). Therefore, higher derivative modes derived from further differentiation of the Newtonian potential v_N^2 with respect to $-a^2$,

$$v_n^2 \equiv \frac{C_n^2}{C_0^2} (-\partial_{a^2})^n v_0^2 = C_n^2 (-\partial_{a^2})^n \left[\frac{a}{\sqrt{r^2 + a^2}} \right] \tag{51}$$

will simply take away the leading constant term from the higher order modes. Therefore, velocity profiles $v_n(r)$ will be quite different from v_0 in this case. Note again that the velocity v_n is induced by the corresponding mass density

$$\mu_n(r) = \frac{C_n^2}{2\pi G} \frac{(2n-1)!!}{2^n} (r^2 + a^2)^{-n-1/2}. \tag{52}$$

One of the advantage of these well-behaved smooth velocity functions is that they can be used as expansion basis for simulation of velocity profiles. Thanks to the linear dependence of the $v_N^2(r)$ in the mass density function $\mu(r)$, one can freely combine any integrable modes of velocity to obtain all possible combinations of integrable models. For example, the model with

$$v^2(r) = v_N^2(r) = \sum_{i,j} v_0^2(r, C_{0i}, a_j) \equiv \sum_{i,j} C_{0i}^2 \left[1 - \frac{a_j}{\sqrt{r^2 + a_j^2}} \right] \tag{53}$$

is integrable and can be shown to be derived by the mass density

$$\mu(r) = \sum_i \mu_0(r, C_{0i}, a_j) \equiv \sum_i \frac{C_{0i}^2}{2\pi G} \frac{1}{\sqrt{r^2 + a_j^2}} \tag{54}$$

with C_{ni} and a_j all constants of parametrization. Here summation over i, j is understood to be summed over all possible modes with different spectrum described by C_{0i} and a_j. This velocity vanishes at $r = 0$ and goes to

$$v^2(r) \to \sum_{i,j} C_{0i}^2 \tag{55}$$

at spatial infinity. One can also add higher derivative terms $v_n^2(r)$ to the velocity profile $v^2(r)$. Since higher derivative velocity v_n vanishes both at $r = 0$ and spatial infinity, these additional terms will not affect the asymptotic behavior of v^2 at $r = 0$. Therefore, one needs to keep at least one zeroth order term in order for v to be compatible with the RC data.

To be more specific, one can consider the model

$$v^2(r) = v_N^2(r) = \sum_{i,j,n} v_n^2(r, C_{ni}, a_j) \equiv \sum_{i,j} C_{0i}^2 \left[1 - \frac{a_j}{\sqrt{r^2 + a_j^2}} \right] + \sum_{k,l,n} C_{nk}^2 (-\partial_{a_l^2})^n \left[\frac{a_l}{\sqrt{r^2 + a_l^2}} \right] \tag{56}$$

which be derived by the mass density

$$\mu(r) = \sum_i \mu_n(r, C_{ni}, a_j) \equiv \sum_{i,j} \frac{C_{0i}^2}{2\pi G} \frac{1}{\sqrt{r^2 + a_j^2}} + \sum_{n,k,l} \frac{C_{nk}^2}{2\pi G} \frac{(2n-1)!!}{2^n} (r^2 + a_l^2)^{-n-1/2} \tag{57}$$

with C_{0i} and a_j are all constants of parameterization. Here the summation over n is to be summed over all $n \geq 1$. The velocity of these models will vanish at $r = 0$ and goes to

$$v^2(r) \to \sum_{i,j} C_{0i}^2 \tag{58}$$

at spatial infinity. Therefore, these models turn out to be good expansion basis of any RC data for Newtonian dynamics.

In practice, one may fit the v^2 in expansion of these modes in order to analyze the RC in basis of of these basis modes. This helps analytical understanding of the spiral galaxies more transparently. The properties of each modes is easy understand because they are integrable. The corresponding coefficients C_{ni} and a_j will determine the contributions of each modes to any galaxies. One will be able to construct tables for spiral galaxies with the corresponding coefficients of each modes. Hopefully, this expansion method originally developed in [23] will provide us a new way to look at the major dynamics of the spiral galaxies.

B. Milgrom model

For the case of Milgrom model (2), the Newtonian velocity v_N and the observed velocity v are related by the following equation

$$v^2(r) = \left[\frac{V_N^4(r) + \sqrt{V_N^8(r) + 4V_N^4(r) g_0^2 r^2}}{2} \right]^{1/2}. \tag{59}$$

Therefore, one can show that a galaxy with a rotation curve given by

$$v_0^4(r) = \frac{C_0^4 a^2}{2(r^2 + a^2)} \left\{ 1 + \sqrt{1 + \frac{4 g_0^2 r^2 (r^2 + a^2)}{C_0^4 a^2}} \right\} \tag{60}$$

is the corresponding velocity induced by the Newtonian velocity

$$v_{N0}^2(r) = \frac{C_0^2 a}{\sqrt{r^2 + a^2}}. \tag{61}$$

Here C_0 and a are some constants of parametrization. Therefore, the corresponding mass density is given by Eq. (39)

$$\mu_0(r) = \frac{C_0^2}{2\pi G} \left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + a^2}} \right].$$

Note that v_0^4 approaches a constant $g_0 C_0^2 a$ at spatial infinity. This asymptotically flat pattern of the velocity is compatible with many observations of the spirals. This Newtonian velocity does not, however, vanish at the origin. Indeed, one can show that $v(0) \to C_0 \neq 0$ Therefore this would not be a good expansion basis for the physical spirals. The non-vanishing behavior of v at $r = 0$ can be secured by considering the refined model (40):

$$v_{N0}^2(r) = C_0^2 \left(\frac{a_1}{\sqrt{r^2 + a_1^2}} - \frac{a_2}{\sqrt{r^2 + a_2^2}} \right)$$

with C_0, $a_1 > a_2$ some constants of parameterizations. The corresponding velocity function v can be shown to be

$$v_0^4(r) = \frac{C_0^4}{2} \left[\frac{a_1}{\sqrt{r^2 + a_1^2}} - \frac{a_2}{\sqrt{r^2 + a_2^2}} \right]^2 \left\{ 1 + \left[1 + \frac{4 g_0^2 r^2 (r^2 + a_1^2)(r^2 + a_2^2)}{C_0^4 \left(a_1 \sqrt{r^2 + a_2^2} - a_2 \sqrt{r^2 + a_1^2} \right)^2} \right]^{1/2} \right\} \tag{62}$$

This new velocity function v is hence induced by the Newtonian velocity (40). In addition, Note that v^4 approaches a constant $g_0 C_0^2(a_1 - a_2)$ at spatial infinity and vanishes at $r = 0$. This agrees with the main feature of the observed asymptotically flat rotation curve of many spirals. As a result, the corresponding mass density of this model is given by Eq. (41):

$$\mu_0(r) = \frac{C_0^2}{2\pi G}\left[\frac{1}{\sqrt{r^2 + a_2^2}} - \frac{1}{\sqrt{r^2 + a_1^2}}\right].$$

In addition, a model with a velocity, in the case of Milgrom model, of the form

$$v_1^4 = \frac{C_1^4 r^4 + \sqrt{C_1^8 r^8 + 4C_1^4 r^6 g_0^2 a^2 (r^2 + a^2)^3}}{2a^2(r^2 + a^2)^3} \tag{63}$$

can be shown to be induced by the Newtonian velocity of the following form given by Eq. (44):

$$v_{N1}^2(r) = \frac{C_1^2 r^2}{a(r^2 + a^2)^{3/2}}.$$

Therefore, this model is derived by the mass density (45):

$$\mu_1(r) = \frac{C_1^2}{2\pi G(r^2 + a^2)^{3/2}}.$$

Note that $v_1^2(r \to \infty) \to C_1\sqrt{g_0/a}$ and $v_1(r = 0) = 0$ in this Milgrom model. In addition, the corresponding Newtonian model also has the same properties: $v_{N1}(r)$ vanishes both at $r = 0$ and $r \to \infty$. Therefore, this model also appears to be a realistic model in agreement with the asymptotically flat rotation curve being observed.

Note again that further differentiation of the Newtonian velocity v_N^2 with respect to $-a^2$ will derive integrable higher derivative models:

$$v_{Nn}^2 \equiv C_n^2(-\partial_{a^2})^n\left[\frac{a}{\sqrt{r^2 + a^2}}\right]. \tag{64}$$

Therefore, this velocity will be derived by the mass density

$$\mu_n(r) = \frac{C_n^2}{2\pi G}\frac{(2n-1)!!}{2^n}(r^2 + a^2)^{-n-1/2}. \tag{65}$$

One of the advantage of these well-behaved smooth velocity functions is that they can be used as expansion basis for simulation of velocity profiles. Thanks to the linear dependence of the $v_N^2(r)$ in the mass density function $\mu(r)$, one can freely combine any integrable modes of velocity to obtain all possible combinations of integrable models. For example, the model with

$$v_N^2(r) = \sum_{i,j} v_{N0}^2(r, C_{0i}, a_j, b_j) \equiv \sum_{i,j} C_{0i}^2\left[\frac{a_j}{\sqrt{r^2 + a_j^2}} - \frac{b_j}{\sqrt{r^2 + b_j^2}}\right] \tag{66}$$

is integrable and can be shown to be derived by the mass density

$$\mu(r) = \sum_i \mu_0(r, C_{0i}, a_j) \equiv \sum_i \frac{C_{0i}^2}{2\pi G}\left[\frac{1}{\sqrt{r^2 + b_j^2}} - \frac{1}{\sqrt{r^2 + a_j^2}}\right] \tag{67}$$

with C_{ni} and a_j all constants of parametrization. The velocity v_N^2 also vanishes at $r = 0$ and goes to

$$v_N^2(r) \to \sum_{i,j} C_{0i}^2\frac{a_j - b_j}{r} \tag{68}$$

at spatial infinity. This will in turn make the corresponding observed Milgrom velocity v^2 approaches the asymptotic velocity $v_\infty^2 \to [\sum_{i,j} C_{0i}^2 g_0(a_j - b_j)]^{1/2}$. One can also adds higher derivative terms $v_{Nn}^2(r)$ to the velocity profile $v_N^2(r)$.

Since higher derivative velocity v_n goes to zero faster than the zero-th derivative term at spatial infinity, these adding will not affect the asymptotic behavior of v^2 at spatial infinity. Therefore, leading order terms will determine the asymptotic value of v. To be more specific, one can consider the model

$$v_N^2(r) = \sum_{i,j,n} v_n^2(r, C_{ni}, a_j) \equiv \sum_{i,j} C_{0i}^2 \left[\frac{a_j}{\sqrt{r^2+a_j^2}} - \frac{b_j}{\sqrt{r^2+b_j^2}} \right] + \sum_{k,l,n} C_{nk}^2 (-\partial_{a_l^2})^n \left[\frac{a_l}{\sqrt{r^2+a_l^2}} \right] \tag{69}$$

which be derived by the mass density

$$\mu(r) = \sum_i \mu_n(r, C_{ni}, a_j) \equiv \sum_{i,j} \frac{C_{0i}^2}{2\pi G} \left[\frac{1}{\sqrt{r^2+b_j^2}} - \frac{1}{\sqrt{r^2+a_j^2}} \right] + \sum_{n,k,l} \frac{C_{nk}^2}{2\pi G} \frac{(2n-1)!!}{2^n} (r^2+a_l^2)^{-n-1/2} \tag{70}$$

with C_{0i} and a_j all constants of parametrization. Note again that the velocity v^2 of these models also vanishes at $r=0$ and goes to

$$v_N^2(r) \rightarrow \sum_{i,j} C_{0i}^2 \frac{a_j - b_j}{r} \tag{71}$$

corresponding to

$$v^2(r) \rightarrow \left[g_0 \sum_{i,j} C_{0i}^2 (a_j - b_j) \right]^{1/2} \tag{72}$$

at spatial infinity. Therefore, these models turn out to be good expansion basis for $v_N^2(r)$ of any RC data for Milgrom models.

In practice, one may convert the RC data from v to v_N following Eq. (59) and then fit the resulting v_N^2 in expansion of these modes in order to analyze the RC in basis of of these basis modes. This helps analytical understanding of the spiral galaxies more transparently. The properties of each modes is easy understand because they are integrable. The corresponding coefficients C_{ni} and a_j will determine the contributions of each modes to any galaxies. One will be able to construct tables for spiral galaxies with the corresponding coefficients of each modes. Hopefully, this expansion method originally developed in [23] will provide us a new way to look at the major dynamics of the spiral galaxies.

C. Famaey and Binney model

For the case of FB model (3), the Newtonian velocity v_N and the observed velocity v are related by the following equation

$$v^2 = \frac{\sqrt{{v_N}^4 + 4g_0 r v_N^2} + v_N^2}{2}. \tag{73}$$

Therefore, one can show that a galaxy with a rotation curve given by

$$v_0^2(r) = \frac{C_0^2 a}{2\sqrt{r^2+a^2}} \left\{ 1 + \left[1 + \frac{4g_0 r\sqrt{r^2+a^2}}{C_0^2 a} \right]^{1/2} \right\} \tag{74}$$

is the corresponding velocity induced by the Newtonian velocity

$$v_{N0}^2(r) = \frac{C_0^2 a}{\sqrt{r^2+a^2}}. \tag{75}$$

Here C_0 and a some constants of parametrization. Therefore, the corresponding mass density is given by Eq. (39)

$$\mu_0(r) = \frac{C_0^2}{2\pi G} \left[\frac{1}{r} - \frac{1}{\sqrt{r^2+a^2}} \right].$$

Note that v_0^4 approaches a constant $g_0 C_0^2 a$ at spatial infinity. This asymptotic flat pattern of the velocity is compatible with many observations of the spirals. This Newtonian velocity does not, however, vanish at the origin. Indeed, one can show that $v_0(0) \to C_0 \neq 0$. Therefore this would not be a good expansion basis for most physical spirals. The non-vanishing behavior of v at $r = 0$ can be secured by considering the refined model (40):

$$v_{N0}^2(r) = C_0^2 \left(\frac{a_1}{\sqrt{r^2 + a_1^2}} - \frac{a_2}{\sqrt{r^2 + a_2^2}} \right)$$

with C_0, $a_1 > a_2$ some constants of parametrization. The corresponding velocity function v can be shown to be

$$v_0^2(r) = \frac{C_0^2}{2} \left[\frac{a_1}{\sqrt{r^2 + a_1^2}} - \frac{a_2}{\sqrt{r^2 + a_2^2}} \right] \left\{ 1 + \left[1 + \frac{4 g_0 r (r^2 + a_1^2)^{1/2} (r^2 + a_2^2)^{1/2}}{C_0^2 \left(a_1 \sqrt{r^2 + a_2^2} - a_2 \sqrt{r^2 + a_1^2} \right)} \right]^{1/2} \right\} \tag{76}$$

This new velocity function v_0 is hence induced by the Newtonian velocity (40). In addition, Note that v_0^4 approaches a constant $g_0 C_0^2 (a_1 - a_2)$ at spatial infinity and vanishes at $r = 0$. This fits the main feature of the asymptotically flat rotation curve of the spirals. As a result, the corresponding mass density of this model is given by Eq. (41):

$$\mu_0(r) = \frac{C_0^2}{2\pi G} \left[\frac{1}{\sqrt{r^2 + a_2^2}} - \frac{1}{\sqrt{r^2 + a_1^2}} \right].$$

In addition, a model with a velocity, in the case of FB model, of the form

$$v_1^2 = \frac{C_1^2 r^2}{2a(r^2 + a^2)^{3/2}} \left[1 + \left[1 + \frac{4 g_0 a (r^2 + a^2)^{3/2}}{C_1^2 r} \right]^{1/2} \right] \tag{77}$$

can be shown to be induced by the Newtonian velocity of the following form given by Eq. (44):

$$v_{N1}^2(r) = \frac{C_1^2 r^2}{a(r^2 + a^2)^{3/2}}.$$

Therefore, this model is derived by the mass density (45):

$$\mu_1(r) = \frac{C_1^2}{2\pi G (r^2 + a^2)^{3/2}}.$$

Note that $v_1^2(r \to \infty) \to C_1 \sqrt{g_0/a}$ and $v_1(r = 0) = 0$ in this FB model. The corresponding Newtonian model also has the same limit: $v_{N1}(r)$ goes to 0 in both $r = 0$ and $r \to \infty$ limits. Therefore, this model appears to be a more realistic model compatible with the flat rotation curve being observed.

Note again that further differentiation of the Newtonian velocity v_N^2 with respect to $-a^2$ will derive integrable higher derivative models:

$$v_{Nn}^2 \equiv C_n^2 (-\partial_{a^2})^n \left[\frac{a}{\sqrt{r^2 + a^2}} \right]. \tag{78}$$

Therefore, this velocity can be shown to be derived from the mass density

$$\mu_n(r) = \frac{C_n^2}{2\pi G} \frac{(2n-1)!!}{2^n} (r^2 + a^2)^{-n-1/2}. \tag{79}$$

One of the advantage of these well-behaved smooth velocity functions is that they can be used as expansion basis for simulation of velocity profiles. Thanks to the linear dependence of the $v_N^2(r)$ in the mass density function $\mu(r)$, one can freely combine any integrable modes of velocity to obtain all possible combinations of integrable models. For example, the model with

$$v_N^2(r) = \sum_{i,j} v_{N0}^2(r, C_{0i}, a_j, b_j) \equiv \sum_{i,j} C_{0i}^2 \left[\frac{a_j}{\sqrt{r^2 + a_j^2}} - \frac{b_j}{\sqrt{r^2 + b_j^2}} \right] \tag{80}$$

is integrable and can be shown to be derived by the mass density

$$\mu(r) = \sum_i \mu_0(r, C_{0i}, a_j) \equiv \sum_i \frac{C_{0i}^2}{2\pi G} \left[\frac{1}{\sqrt{r^2 + b_j^2}} - \frac{1}{\sqrt{r^2 + a_j^2}} \right] \tag{81}$$

with C_{ni} and $a_j > b_j$ are all constants of parameterizations. The velocity will vanish at $r = 0$ and goes to

$$v_N^2(r) \to \sum_{i,j} C_{0i}^2 \frac{a_j - b_j}{r} \tag{82}$$

at spatial infinity. This will in turn make the corresponding observed FB velocity v approach the asymptotic velocity $v_\infty^2 \to [\sum_{i,j} C_{0i}^2 g_0(a_j - b_j)]^{1/2}$. One can also adds higher derivative terms $v_{Nn}^2(r)$ to the velocity profile $v_N^2(r)$. Since higher derivative velocity v_n goes to zero faster than the zero-th derivative term at spatial infinity, these adding will not affect the asymptotic behavior of v^2 at spatial infinity. Therefore, the leading order terms will determine the asymptotic behavior of the RC.

To be more specific, one can consider the model

$$v_N^2(r) = \sum_{i,j,n} v_n^2(r, C_{ni}, a_j) \equiv \sum_{i,j} C_{0i}^2 \left[\frac{a_j}{\sqrt{r^2 + a_j^2}} - \frac{b_j}{\sqrt{r^2 + b_j^2}} \right] + \sum_{n,k,l} C_{nk}^2 (-\partial_{a_l^2})^n \left[\frac{a_l}{\sqrt{r^2 + a_l^2}} \right] \tag{83}$$

which be derived by the mass density

$$\mu(r) = \sum_i \mu_n(r, C_{ni}, a_j) \equiv \sum_{i,j} \frac{C_{0i}^2}{2\pi G} \left[\frac{1}{\sqrt{r^2 + b_j^2}} - \frac{1}{\sqrt{r^2 + a_j^2}} \right] + \sum_{n,k,l} \frac{C_{nk}^2}{2\pi G} \frac{(2n-1)!!}{2^n} (r^2 + a_l^2)^{-n-1/2} \tag{84}$$

with C_{0i} and a_j all constants of parameterization. The velocity of these models vanishes at $r = 0$ and goes to

$$v_N^2(r) \to \sum_{i,j} C_{0i}^2 \frac{a_j - b_j}{r}, \tag{85}$$

corresponding to

$$v^2(r) \to \left[g_0 \sum_{i,j} C_{0i}^2 (a_j - b_j) \right]^{1/2} \tag{86}$$

at spatial infinity. Therefore, these models turn out to be good expansion basis for v_N^2 of any RC data for FB models.

In practice, one may convert the RC data from v to v_N following Eq. (73) and then fit the resulting v_N^2 in expansion of these modes in order to analyze the RC in basis of of these basis modes. This helps analytical understanding of the spiral galaxies more transparently. The properties of each modes is easy understand because they are integrable. The corresponding coefficients C_{ni} and a_j will determine the contributions of each modes to any galaxies. One will be able to construct tables for spiral galaxies with the corresponding coefficients of each modes. Hopefully, this expansion method originally developed in [23] will provide us a new way to look at the major dynamics of the spiral galaxies.

IV. COMPACT AND REGULAR EXPRESSION

In order to put the integral in a numerically accessible form, Eq. (28) for $\mu(r)$ can be written as a more compact form with a compact integral domain $x \in [0, 1]$:

$$\begin{aligned} \mu(r) = & \frac{1}{\pi^2 G r} \\ & \times \left[\int_0^1 \partial_x [v_N^2(rx)] \, K(x) dx - \int_0^1 \partial_y [v_N^2(\frac{r}{y})] \, K(y) y dy \right]. \end{aligned} \tag{87}$$

Here one has replaced $x = r'/r$ and $y = r/r'$ in above integral. One of the advantage of this expression is the numerical analysis involves only a compact integral domain $0 \leq x \leq 1$ instead of a open and infinite domain $0 \leq r \to \infty$ domain. Even most integral vanish quickly enough without bothering the large r domain, the compact expression will make both the numerical and analytical implication more transparent to access.

Note that the function $K(x)$ diverges at $x = 1$. It is, however, easy to show that $K(x)dx \to 0$ near $x = 1$. Usually, one can manually delete the negligible integration involving the elliptical function $K(x \to 1)$ to avoid computer break-down due to the apparent singularity.

It will be, however, easier for us to perform analytic and/or numerical analysis with an equation that is free of any apparently singular functions in the integrand. Indeed, this can be done by transforming the singular elliptic function $K(x)$ to regular elliptic function $E(x)$. The advantage of this transformation will be used to evaluate approximate result in the next sections. Therefore, one will try to convert the apparently singular function $K(x)$ into a singular free function $E(x)$ by performing some proper integration-by-part.

One will need a few identities satisfied by the elliptic functions E and K. Indeed, it is straightforward to show that $E(x)$ and $K(x)$ satisfy the following equations that will be useful in converting the integrals into more accessible form:

$$
\begin{aligned}
x(1-x^2)K''(x) + (1-3x^2)K'(x) - xK(x) &= 0, \quad (88)\\
x(1-x^2)E''(x) + (1-x^2)E'(x) + xE(x) &= 0, \quad (89)
\end{aligned}
$$

and also

$$
\begin{aligned}
xE'(x) + K(x) &= E(x), \quad (90)\\
E'(x) + \frac{x}{1-x^2}E(x) &= K'(x), \quad (91)\\
xK'(x) + K(x) &= \frac{1}{1-x^2}E(x). \quad (92)
\end{aligned}
$$

Note that Eq. (90) can be derived directly from differentiating the definition of the elliptic integrals Eq.s (23-24). In addition, Eq.s (91-92) can also be derived with the help of the Eq. (89).

When one is given a set of data as a numerical function of $v_N(rx)$, it is much easier to compute $dv_N(k = rx)/dk$ instead of $\partial_x v_N(rx)$. Therefore, one will need the following converting formulae:

$$\partial_x[v_N^2(rx)] = r[v_N^2(rx)]',$$

$$\partial_y[v_N^2(\frac{r}{y})] = -\frac{r}{y^2}[v_N^2(\frac{r}{y})]'.$$

Therefore, one is able to write the Eq. (87) as:

$$\mu(r) = \frac{1}{\pi^2 G}\left[\int_0^1 dv_N^2(rx)\ K(x)dx + \int_0^1 dv_N^2(\frac{r}{y})\ K(y)y^{-1}dy\right] \quad (93)$$

Here $dv_N^2(r) \equiv \partial_r[v_N^2(r)] \equiv [v_N^2(r)]'$ with $'$ denoting the differentiating with the argument r, or rx.

Note that the part involving the integral with $dv_N^2(r/y)$ is related to the information in the region $r' \geq r$. Here r is the point of the derived information such as $\mu(r)$, and r' is the source point of observation $v(r')$ in the integrand. Therefore, this part with source function r/y contains information exterior to the target point r. On the contrary, the source term with function of rx represents the information interior to the target point r.

Most of the time, the source information beyond certain observation limit $r' = R$ becomes un-reliable or unavailable due to the precision limit of the observation instruments. One will therefore need to manually input the missing data in order to make the integration result free of any singular contributions due to the boundary effect. One will come back to this point in section IV.

In addition, Eq. (93) can be used to derive the total mass distribution $M(r)$ of the spiral galaxy via the following equation:

$$M(r) = 2\pi\int_0^r r'dr'\ \mu(r'). \quad (94)$$

Note that the velocity function v_N shown previously in this paper is the rotation velocity needed to work with the total mass of the system in the Newtonian dynamics. One can derive the velocity $v(r)$ that works with the dynamics of MOND with the relation given by Eq. (1) and (2). In short, $v \to v_N$ in the limit of the Newtonian dynamics.

Throughout the rest of this paper, we will discuss the application of these formulae both in the case of the Newtonian dynamics with dark matter and in the case of MOND. Therefore, we will first derive the velocity function $v_N(r)$ from $v(r)$ in the case of MOND. As mentioned above, they follows the relation given by Eq. (1) and (2).

Two different models will be studied later:

Case I: Milgrom model

Indeed, if one has $g(r) = v^2(r)/r$ in the case of Milgrom model, one can show that

$$v_N^2(r) = \frac{v^4(r)}{\sqrt{v^4(r) + g_0^2 r^2}} \tag{95}$$

In dealing with the exterior part involving r_0/y, one has to compute $dv_N^2(r)$ at large r. By assuming $v(r) \to v_R \equiv v(r = R)$, one can show that:

$$dv_N^2(r \geq R) \to -\frac{v_R^4 g_0^2 r}{(v_R^4 + g_0^2 r^2)^{3/2}}. \tag{96}$$

Here R is the radius of the luminous galactic boundary. Mostly, the flatten region of RC becomes manifest beyond $r \geq R$. In addition, $v_N(rx)$ and $v_N(r/y)$ take the following form:

$$v_N^2(rx) = \frac{v^4(rx)}{\sqrt{v^4(rx) + g_0^2 r^2 x^2}}, \tag{97}$$

$$v_N^2(\frac{r}{y}) = \frac{v^4(r/y)y}{\sqrt{v^4(r/y)y^2 + g_0^2 r^2}}. \tag{98}$$

Therefore, the surface mass density $\mu(r)$ can be written as, with the velocity $v_N(r)$ given above,

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty \partial_{r'} [\frac{v^4}{\sqrt{v^4 + g_0^2 r'^2}}] \; H(r, r') dr' \tag{99}$$

in the case of Milgrom model.

Case II: FB model

Similarly, if one has $g(r) = v^2(r)/r$ in the case of FB model, one can show that

$$v_N^2(r) = \frac{v^4(r)}{v^2(r) + g_0 r} \tag{100}$$

By assuming $v(r) \to v_R \equiv v(r = R)$, one can also show that:

$$dv_N^2(r \geq R) \to -\frac{v_R^4 g_0}{(v_R^2 + g_0 r)^2}. \tag{101}$$

In addition, $v_N(rx)$ and $v_N(r/y)$ take the following form:

$$v_N^2(rx) = \frac{v^4(rx)}{v^2(rx) + g_0 rx}, \tag{102}$$

$$v_N^2(\frac{r}{y}) = \frac{v^4(r/y)y}{v^2(r/y)y + g_0 r}. \tag{103}$$

Therefore, the surface mass density $\mu(r)$ can be written as, with the velocity $v_N(r)$ given above,

$$\mu(r) = \frac{1}{2\pi G} \int_0^\infty \partial_{r'} [\frac{v^4}{v^2 + g_0 r'}] \; H(r, r') dr' \tag{104}$$

in the case of FB model. We will try to estimate the exterior contribution of these two models shortly.

As mentioned above, it is easier to handle the numerical evaluation involving regular function $E(x)$ instead of the singular function $K(x)$. Therefore, one can perform an integration-by-part and convert the integral in Eq. (87) into an integral free of singular function $K(x)$. The result reads, with $\mu(r) = \Theta(r)/(\pi^2 G r)$,

$$\Theta(r) = \int_0^1 \frac{V_N^2(r/x) - xV_N^2(rx)}{1-x^2} E(x)dx - \int_0^1 E'(x)V_N^2(rx)dx \tag{105}$$

The last term on the right hand side of above equation can be integrated by part again to give

$$\begin{aligned}\Theta(r) &= \int_0^1 \frac{V_N^2(r/x) - xV_N^2(rx)}{1-x^2} E(x)dx \\ &+ \int_0^1 E(x)\partial_x V_N^2(rx)dx - V_N^2(r).\end{aligned} \tag{106}$$

Hence one has

$$\begin{aligned}\mu(r) &= \frac{1}{\pi^2 G r}[\int_0^1 \frac{V_N^2(r/x) - xV_N^2(rx)}{1-x^2} E(x)dx \\ &+ \int_0^1 E(x)\partial_x V_N^2(rx)dx - V_N^2(r)].\end{aligned} \tag{107}$$

Note again that the integral involving $v_N(rx)$ carries the information $r' \le r$ while the integral with $v_N(r/x)$ represents the contribution from $r' > r$ by the fact that $0 \le x \le 1$. As promised, one has transformed the singular K function into the regular E function.

Although there are still singular contribution like $1/(1-x^2)$ in the integrand, it is easier to handled since we know these functions better than K function. This is because we only know a formal definition of this function via a set of definitions. Even we have a rough picture about the form of $K(x)$ and $K'(x)$. Numerical and analytical analysis could be difficult as compared to dealing with the more well-known function like $1/(1-x^2)$. We will show explicitly one of the advantage of this equation in next section when one is trying to estimate the contribution from a model describing the missing part of observation.

V. CONTRIBUTION FROM THE ASYMPTOTIC REGION

In practice, measurement in far out region is normally difficult and unable to provide us with reliable information beyond the sensitivity limit of the observation instrument. One often can only measure energy flux and rotation curve within a few hundred *kpc* from the center of the galaxy. Beyond that scale of range, signal is normally too weak to obtain any reliable data. Therefore, one has to rely on various models to interpolate the required information further out.

It it known that, contrary to the spherically system, exterior mass does contribute to the inner region. Therefore, it is important to estimate the exterior contribution carefully with various models. In this section, we will study a velocity model with a flat asymptotic form and its contribution to the inner part in both Newtonian dynamics and MOND cases. One of the purpose of doing this is to demonstrate the advantage of the regular function formulae one derived earlier.

For a highly flatten galaxy, formulae obtained earlier in previous section has been shown to be a very useful tool to predict the dynamics of spiral galaxies. It is also a good tool for error estimation. Possible deviation due to the interpolating data can be estimated analytically more easily with the equations involving only regular elliptic function $E(x)$. Note again that another advantage of Eq. (107) is that the r-dependence of the mass density $\mu(r)$ has been extracted to the function v_N. This will make the analytical analysis easier too.

Assuming that the observation data v is only known for the region $r \le R$, the following part of Eq. (107)

$$\delta\mu(r \le R) = \frac{1}{\pi^2 G r} \int_0^{r/R} dx \left[\frac{v_N^2(r/x)}{1-x^2} E(x) \right] \tag{108}$$

represents the contribution of $\mu(r \le R)$ from the unavailable data $v_N(r')$ beyond the point $r = R$. To be more precise, one can write $\mu(r \le R) = \mu_<(r) + \delta\mu(r)$ with the mass density $\mu_<(r)$ being contributed solely from the

$v_N(r') = v_N(r/x)$ data between $0 \le r' \le R$ or equivalently $r/R \le x \le 1$. Explicitly, $\mu_<(r)$ can be expressed as

$$\begin{aligned}\mu_<(r) &= \frac{1}{\pi^2 G r}[\int_{r/R}^1 \frac{V_N^2(r/x)}{1-x^2}E(x)dx - \int_0^1 \frac{xV_N^2(rx)}{1-x^2}E(x)dx \\ &+ \int_0^1 E(x)\partial_x V_N^2(rx)dx - V_N^2(r)]. \end{aligned} \tag{109}$$

As mentioned above, one will need a model for the unavailable data to estimate the contribution shown in Eq. (108). We will show that a simple cutoff with $v(r > R) = 0$ will introduce a logarithmical divergence to the surface density $\mu(r = R)$. The divergence is derived from the singular denominator $1-x^2$ in Eq. (107). The factor $1/(1-x^2)$ diverges at $x = 1$ or equivalently $r' = r$. A smooth velocity function $v_N(r)$ connecting the region $R-\epsilon < r < R+\epsilon$ is required to make the combination $[V_N^2(r/x) - xV_N^2(rx)]/(1-x^2)$ regular at $x = 1$. Here ϵ is an infinitesimal constant.

Case I: Milgrom model

Let us study first the case of Milgrom model. Since most spiral galaxies has a flat rotation curve $v(r \gg R) \to v_R$ with a constant velocity v_R. For our purpose, let us assume that $v_R = v(r = R)$ for simplicity. Hence the deviation (108) can be evaluated accordingly. Note again that this simple model agrees very well with many known spirals.

Note first that the Newtonian velocity $v_N(r)$ is given by Eq.s (97) and (98) with $v(r > R) = V_R$. After some algebra, one can show that

$$\delta\mu_M(r < R) = \frac{1}{\pi^2 G r}\int_0^{r/R} dx \left[\frac{v_N^2(r/x)}{1-x^2}\right]E(x) \tag{110}$$

$$= \frac{E_M}{\pi^2 G r}\int_0^{r/R} dx \left[\frac{v_N^2(r/x)}{1-x^2}\right] \tag{111}$$

$$\begin{aligned}&= \frac{E_M v_R^4}{\pi^2 G r\sqrt{v_R^4 + g_0^2 r^2}} \\ &\times \left[\ln \frac{r\sqrt{v_R^4 + g_0^2 R^2} + R\sqrt{v_R^4+g_0^2 r^2}}{(g_0 r + \sqrt{v_R^4 + g_0^2 r^2})\sqrt{R^2 - r^2}}\right]\end{aligned} \tag{112}$$

Note that $\pi/2 \ge E(x) \ge 1$ is a monotonically decreasing function with a rather smooth slope. The rest of the integrand is also positive definite. Therefore, one can evaluate the integral by applying the mean value theorem for the integral (110) with $E_M \equiv E(x = x_M)$ the averaged value of $E(x)$ evaluated at x_M somewhere in the range $0 \le x_M \le 1$.

Case II: FB model.

Let us study instead the case of FB model. Let us also assume that $v_R = v(r = R)$ for simplicity. Note first that the Newtonian velocity $v_N(r)$ is given instead by Eq.s (102) and (103) with $v(r > R) = V_R$. After some algebra, one can show that

$$\begin{aligned}\delta\mu_F(r < R) &= \frac{E_F}{\pi^2 G r}\int_0^{r/R} dx \left[\frac{v_N^2(r/x)}{1-x^2}\right] \\ &= \frac{E_F v_R^4}{\pi^2 G r}\int_0^{r/R} dx \left[\frac{1}{(1-x^2)(v_R^2 + g_0 r/x)}\right] \equiv \frac{E_F v_R^2}{\pi^2 G r} I\end{aligned} \tag{113}$$

with

$$I = \int_0^{r/R} dx \left[\frac{x}{(1-x^2)(x + g_0 r/v_R^2)}\right] \tag{114}$$

Note that $\pi/2 \ge E(x) \ge 1$ is a monotonically decreasing function with a rather smooth slope. The rest of the integrand is also positive definite. Therefore, one can evaluate the integral by applying the mean value theorem for the integral (113) with $E_F \equiv E(x = x_F)$ the averaged value of $E(x)$ evaluated at x_F somewhere in the range $0 \le x_F \le 1$. After some algebra, one can show that

$$\delta\mu_F(r < R) = \frac{E_F v_R^4}{2\pi^2 G r}\left[\frac{1}{v_R^2 - g_0 r}\ln(1 + r/R) - \frac{1}{v_R^2 + g_0 r}\ln(1 - r/R) - \frac{2g_0 r}{v_R^4 - g_0^2 r^2}\ln[1 + v_R^2/(g_0 R)]\right]. \tag{115}$$

Case III: Newtonian case.

Similarly, one can also evaluate the mass density in the case of Newtonian dynamics with dark matter in need. Let us assume $v_N(r \geq R) = v_R$ for simplicity again. As a result the deviation of mass density required to generate the rotation curve $v_N(r) = v(r)$ can be shown to be

$$\begin{aligned}\delta\mu_N(r < R) &= \frac{E_N}{\pi^2 G r} \int_0^{r/R} dx \left[\frac{v_N^2(r/x)}{1 - x^2} \right] \\ &= \frac{E_N v_R^2}{2\pi^2 G r} \ln \frac{R + r}{R - r} \end{aligned} \tag{116}$$

after some algebra. Note that, in deriving above equation, one also applies the mean value theorem with $E_N \equiv E(x = x_N)$ the averaged value of $E(x)$ evaluated at x_N somewhere in the range $0 \leq x_N \leq 1$.

To summarize, one has shown clearly with a simple model for the asymptotic rotation velocity that formulae with regular E function appears to be easier for analysis. This is mainly due to the fact that $E(x)$ is smoothly and monotonically decreasing function within the whole domain $x \in [0, 1]$. In most cases, mean value theorem is very useful in both numerical and analytical evaluations.

In addition, one notes that the mild logarithm divergent terms appeared in the above final results are due to the cut-off at $r = R$. A negative and equal contribution from the interior data will cancel this singularity at $r = R$. To be more precisely, if we turn off the v function abruptly starting the point $r = R$ by ignoring the exterior region contribution, a logarithmic divergence will show up at $r = R$ accordingly. The appearance of the logarithm divergence also emphasize that the boundary condition of these physical observables at $r = R$ should be taken care of carefully to avoid these unphysical divergences. In practice, one normally adds a quickly decreasing $v_N(r > R)$ to account for the missing pieces of information and to avoid this singularity. Numerical computation may, however, bring up a small peak near the boundary $r = R$ if the matching of v_N at the cutoff is not smooth enough. One will also show that similar singular behavior also appears in the final expression of the velocity function derived from a given data of mass distribution in next section. Evidence also shows that exterior contribution should be treated carefully in order to provide a meaningful fitting result.

In order to compare the difference of $\delta\mu$ for different models, we find it is convenient to write $B \equiv v_R^2/(g_0 R)$ and $s \equiv r/R$ such that A and r' both become dimensionless parameters. Note that $B \sim 1.1$ if we take $v_R \sim 250\, km/s$ and $R \sim 4.97 \times 10^4\, ly$ from the data of Milky Way. Therefore, $B \sim 1.1$ is typically a number slightly larger than 1. Hence, one can write above equations as

$$\delta\mu_M(s < 1) = \frac{E_M B^2 g_0}{\pi^2 G s \sqrt{B^2 + s^2}} \left[\ln \frac{s\sqrt{B^2 + 1} + \sqrt{B^2 + s^2}}{(s + \sqrt{B^2 + s^2})\sqrt{1 - s^2}} \right], \tag{117}$$

$$\delta\mu_F(s < 1) = \frac{E_F B^2 g_0}{2\pi^2 G s} \left[\frac{\ln(1 + s)}{B - s} - \frac{\ln(1 - s)}{B + s} + \frac{2s\ln(1 + B)}{s^2 - B^2} \right], \tag{118}$$

$$\delta\mu_N(s < 1) = \frac{E_N B g_0}{2\pi^2 G s} \ln \frac{1 + s}{1 - s}. \tag{119}$$

In addition, one can estimate the deviation $\delta\mu$ at small s where $s \ll 1$, or $r \ll R$. The leading terms read:

$$\delta\mu_M(s \ll 1) = \frac{E_M g_0}{\pi^2 G} \left[\sqrt{B^2 + 1} - 1 \right] + \mathrm{O}(s), \tag{120}$$

$$\delta\mu_F(s \ll 1) = \frac{E_F g_0}{\pi^2 G} \left[B - \ln(B + 1) \right] + \mathrm{O}(s), \tag{121}$$

$$\delta\mu_N(s \ll 1) + \mathrm{O}(s). = \frac{E_N g_0}{\pi^2 G} B \tag{122}$$

at small r. Note that $g_0/G \sim 0.18$. Therefore, the most important contribution from the exterior contribution is near the boundary at $r = R$. Our result shows that special care must be taken near the boundary of available data. Appropriate matching data beyond this boundary is needed to eliminated the naive logarithm divergence. The compact expression also made reliable estimation of the deviation possible.

VI. GRAVITATIONAL FIELD DERIVED FROM A GIVEN MASS DENSITY

One can measure the flux from a spiral galaxy and try to obtain the mass density with the $M/L =$ constant law [24]. Even the M/L law is more or less an empirical law, it does help us with a fair estimate of the mass density distribution. We will focus again on the physics of a highly flattened spiral galaxy. Once the mass density function

is known, for the range $0 \le r \le R$, one can also compute the gravitational field $g_N(r)$ from a given mass density function $\mu(r)$. Once the function $g_N(r)$ is derived, one can derive $g(r)$ following the relation given by Eq.s (1) and (2) in the case of MOND.

Indeed, one can show that $g_N(r)$ is given by

$$g_N(r) = 2\pi G \int_0^\infty kdk \int_0^\infty r'dr' \; \mu(r') J_0(kr') J_1(kr) \tag{123}$$

from the Eq.s (6) and (11). Therefore one has

$$g_N(r) = 2\pi G \int_0^\infty r'dr' \; \mu(r') H_1(r, r') \tag{124}$$

with

$$H_1(r, r') = \int_0^\infty kdk J_1(kr) J_0(kr') = -\partial_r H(r, r'). \tag{125}$$

By differentiating Eq. (18), one can further show that $H_1(r, r')$ becomes

$$H_1(r, r') = \frac{2}{\pi r r'} \left[K(\frac{r}{r'}) - \frac{r'^2}{r'^2 - r^2} E(\frac{r}{r'}) \right] \text{ for } r < r' \tag{126}$$

$$= \frac{2}{\pi (r^2 - r'^2)} E(\frac{r'}{r}) \text{ for } r > r'. \tag{127}$$

Note that one has used the differential equations obeyed by E and K shown in Eq.s (90)-(92). Following the method shown in section IV, one can rewrite the equation as

$$g_N(r) = 4G \int_0^1 dx \left[\mu(rx) \frac{xE(x)}{1 - x^2} + \mu(r/x) [\frac{K(x)}{x^2} - \frac{E(x)}{x^2(1 - x^2)}] \right] \tag{128}$$

after some algebra. In order to eliminate the singular function $K(x)$, we can also convert $K(x)$ into regular function $E(x)$ following similar method. The result is

$$g_N(r) = 4G \int_0^1 dx \left[\mu(rx) \frac{xE(x)}{1 - x^2} - \mu(r/x) [\frac{E(x)}{1 - x^2} + \frac{E'(x)}{x}] \right]. \tag{129}$$

With an integration-by-part, one can convert $E'(x)$ to a regular function $E(x)$. The result is

$$g_N(r) = 2G\mu_*(r) - 2\pi G\mu(r) + 4G \int_0^1 dx \left[\frac{E(x)}{x^2(1 - x^2)} [x^3 \mu(rx) - \mu(r/x)] - \frac{E(x) r \mu'(r/x)}{x^3} \right]. \tag{130}$$

Here $\mu_*(r) \equiv \lim_{x \to 0} 2\mu(r/x)/x$. If $\mu(r \to \infty)$ goes to 0 faster than the divergent rate of r, $\mu_*(r)$ will vanish or remain finite. For example, if $\mu(r > R) = 2\mu_R R r/(R^2 + r^2)$, one can show that $\mu_*(r) = 4\mu_R R/r$. Here $\mu_R \equiv \mu(r = R)$. We will be back with this model in a moment.

Similar to the argument shown in section IV, one can show that the terms with $\mu(r/x)$ in Eq. (130) will contribute

$$\delta g_{N0}(r \le R) = -4G \int_0^{r/R} dx E(x) \left[\frac{\mu(r/x)}{x^2(1 - x^2)} + \frac{r\mu'(r/x)}{x^3} \right] \tag{131}$$

to the function $g_N(r)$ due to the unavailable data $\mu(r > R)$. Note, however, that part of the exterior region contribution has been evaluated via the integration-by-part process in deriving Eq. (130) involving $v_N(r/x)$ and $E'(x)$. Therefore,

one should start with the complete version (129) in evaluating the deviation of g_N due to the exterior part. Hence one should have

$$\delta g_N(r \leq R) = -4G \int_0^{r/R} dx \mu(\frac{r}{x}) \left[\frac{E(x)}{1-x^2} + \frac{E'(x)}{x} \right]. \tag{132}$$

For simplicity, we will assume the following form of mass distribution in the region $r > R$,

$$\mu(r > R) = \mu_R \frac{2Rr}{R^2 + r^2} \tag{133}$$

with $\mu_R \equiv \mu(r = R)$. Note that continuity of $\mu(r = R)$ across the matching point $r = R$ is managed to remain valid in this model. Moreover, $\mu(r \to \infty) \to 0$ is expected to hold for the luminous mass of the spiral galaxies. After some algebra, one can show that

$$\delta g_N(r) = -4G\mu_R \frac{R}{r} \left[E(\frac{r}{R}) - \pi + A(r) \right] \tag{134}$$

with $A(r)$ given by the integral

$$\begin{aligned} A &= \int_0^{1/b} dy \left[\frac{1}{(1-y)(1+by)} + \frac{2b}{(1+by)^2} \right] E(\sqrt{y}) && (135) \\ &= E_1 \int_0^{1/b} dy \left[\frac{1}{(1-y)(1+by)} + \frac{2b}{(1+by)^2} \right] && (136) \end{aligned}$$

with the first two terms in Eq. (134) the contribution from integration-by-part of $E'(x)$. Here one has defined $y = x^2$ and $b = R^2/r^2$ for convenience.

In addition, one also applies the mean value theorem to the integral (135) by noting again that (i) $\pi/2 \geq E(x) \geq 1$ is a monotonically decreasing function with a rather smooth slope, and (ii) the integrand is a positive function throughout the integration range. Therefore, the integral $A(r)$ can be evaluated by applying the mean value theorem with $E_1 \equiv E(x = x_1)$ the averaged value of $E(x)$ evaluated at x_1 somewhere in the range $0 \leq x_1 \leq 1$.

The remaining integral in A can be evaluated in a straightforward way and one finally has

$$\begin{aligned} &\delta g_N(r) = && (137) \\ &-4G\mu_R \frac{R}{r} \left[E_1 [1 + \frac{r^2}{R^2 + r^2} \ln \frac{2R^2}{R^2 - r^2}] + E(\frac{r}{R}) - \pi \right]. \end{aligned}$$

This is the contribution to the Newtonian field $g_N(r \leq R)$ due to the unknown exterior mass contribution. Note that similar singularity also appears at $\delta g_N(r = R)$ which means that a careful treatment in modelling the unknown exterior mass is needed.

Case I: Milgrom model.

For the case of Milgrom model, one can show that

$$g(r) = \sqrt{\frac{g_N^2(r) + \sqrt{g_N^4(r) + 4g_N^2(r) g_0^2}}{2}}. \tag{138}$$

Therefore, the deviation $\delta g^2(r)$ can be shown to be

$$\delta g^2(r) = \left[\frac{1}{2} + \frac{g_N^2(r) + 2g_0^2}{2\sqrt{g_N^4(r) + 4g_N^2(r) g_0^2}} \right] \delta g_N^2(r) \tag{139}$$

by solving the algebraic equations (1)-(2).

Case II: FB model.

For the FB model, one can show that

$$g = \frac{\sqrt{{g_N}^2 + 4g_0 g_N} + g_N}{2}. \tag{140}$$

Therefore, the deviation $\delta g^2(r)$ can be shown to be

$$\delta g(r) = \frac{1}{2}\left[1 + \frac{g_N(r) + 2g_0}{\sqrt{g_N^2(r) + 4g_N(r)g_0}}\right]\delta g_N(r) \tag{141}$$

by solving the algebraic equations (140).

Therefore, the deviation $\delta g(r)$ can be evaluated from above equation to the first order in $\delta g_N(r)$ with all $g_N(r)$ replaced by $g_{N<}(r)$. Here $g_{N<}(r)$ is defined as the contribution of interior mass to the Newtonian field $g_N(r)$, namely,

$$g_N(r \leq R) = g_{N<}(r) + \delta g_N(r). \tag{142}$$

To summarize again, one has shown that the formulae with regular E function appears to be helpful in deriving the gravitational field strength for numerical and analytical purpose.

VII. CONCLUSION

We have reviewed briefly how to obtain the surface mass density $\mu(r)$ from a given Newtonian gravitational field g_N with the help of the elliptic function $K(r)$ in this paper. The integral involving the Bessel functions is derived in detailed for heuristical reasons in this paper too. In addition, a series of integrable model in the case of MOND is also presented in this paper.

One has also converted these formulae into a simpler compact integral making numerical integration more accessible and analytical estimate possible in this paper. The apparently singular elliptic function $K(r)$ is also converted to combinations of regular elliptic function $E(r)$ by properly managed integration-by-part.

As a physical application, one derives the interior mass contribution $\mu(r < R)$ from the possibly unreliable data $v(r > R)$ both in the cases of MOND and in the Newtonian dynamics. Detailed results are presented for Milgrom model and FB model for the case of MOND. In particular, the singularity embedded in these formulae are shown to be a delicate problem requiring great precaution. Similarly, one also tries to derive similar results for the corresponding g_N from a given $\mu(r)$. One also presents a simple model of exterior mass density $\mu(r > R)$ as a simple demonstration. The corresponding result in the theory of MOND is also presented in this paper.

In section III, one has studied many solvable models in details. Analysis is generalized to Newtonian models, Milgrom models as well as the FB models. In practice, may convert the RC data from v to v_N following the transformation formula of either Milgrom model or FB model. For the case of Newtonian model, the RC data gives exactly the Newtonian velocity, namely, $v = v_N$. One can then fit the resulting v_N^2 in expansion of these modes in order to analyze the RC in basis of of these basis modes. This helps analytical understanding of the spiral galaxies more transparently. The properties of each modes can be easily understood because they are integrable. The corresponding coefficients C_{ni} and a_j will determine the contributions of each modes to any galaxies. One will then be able to construct tables for spiral galaxies with the corresponding coefficients of each modes. Hopefully, this expansion method will provide us a new way to look at the major dynamics of the spiral galaxies. Analytic approach to the dynamics of highly flattened galaxies

In summary, the compact expressions (108) and (130) have been shown in this paper to be useful in the estimate of the mass density $\mu(r < R)$ and $g(r < R)$ from the exterior data at $r' > R$. Explicit models are presented in this paper.

One also presents a more detailed derivation involving the definition of the elliptic functions E and K. Various useful formulae are also presented for heuristic purpose.

One also focuses on the application in the case of modified Newtonian dynamics for two different successful models: the Milgrom model and the FB model. The theory of MOND appears to be a very successful model representing possible alternative to the dark matter approach. Nonetheless, MOND could also be the collective effect of some quantum fields under active investigations [10, 22]. The method and examples shown in this paper should be of help in resolving the quested puzzle.

Acknowledgments

Acknowledgments This work is supported in part by the National Science Council of Taiwan. We thank

professor Zhao H.S. for helpful suggestions and comments.

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